5.2 Mean Value Theorem

What you will learn about . . .

- Mean Value Theorem
- Physical Interpretation
- Increasing and Decreasing Functions
- Other Consequences

and why . . .

The Mean Value Theorem is an important theoretical tool to connect the average and instantaneous rates of change.

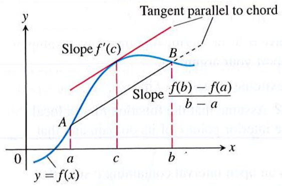
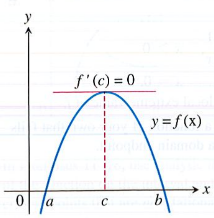


Figure 5.10 Figure for the Mean Value Theorem.

Rolle's Theorem

The first version of the Mean Value Theorem was proved by French mathematician Michel Rolle (1652–1719). His version had $f(a) = f(b) = 0$. and was proved only for polynomials, using algebra and geometry.



Rolle distrusted calculus and spent most of his life denouncing it. It is ironic that he is known today only for an unintended contribution to a field he tried to suppress.

Mean Value Theorem

The Mean Value Theorem connects the average rate of change of a function over an interval with the instantaneous rate of change of the function at a point within the interval. Its powerful corollaries lie at the heart of some of the most important applications of the calculus.

The theorem says that somewhere between points A and B on a differentiable curve, there is at least one tangent line parallel to chord AB (Figure 5.10).

THEOREM 3 Mean Value Theorem for Derivatives

If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one point c in (a, b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The hypotheses of Theorem 3 cannot be relaxed. If they fail at even one point, the graph may fail to have a tangent parallel to the chord. For instance, the function $f(x) = |x|$ is continuous on $[-1, 1]$ and differentiable at every point of the interior $(-1, 1)$ except $x = 0$. The graph has no tangent parallel to chord AB (Figure 5.11a). The function $g(x) = \text{int}\,(x)$ is differentiable at every point of $(1, 2)$ and continuous at every point of $[1, 2]$ except $x = 2$. Again, the graph has no tangent parallel to chord AB (Figure 5.11b).

The Mean Value Theorem is an *existence theorem.* It tells us the number c exists without telling how to find it. We can sometimes satisfy our curiosity about the value of c but the real importance of the theorem lies in the surprising conclusions we can draw from it.

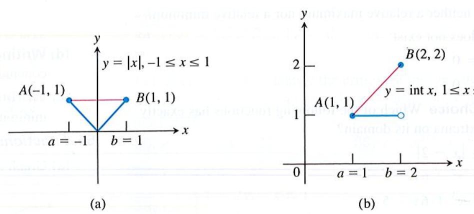


Figure 5.11 No tangent parallel to chord AB.

EXAMPLE 1 Exploring the Mean Value Theorem

Show that the function $f(x) = x^2$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 2]$. Then find a solution c to the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

on this interval.

continued

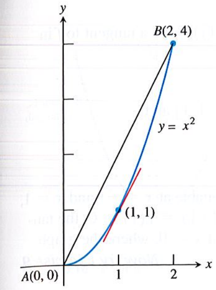


Figure 5.12 (Example 1)

SOLUTION

The function $f(x) = x^2$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem guarantees a point c in the interval $(0, 2)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$2c = \frac{f(2) - f(0)}{2 - 0} = 2 \quad f'(x) = 2x$$

$$c = 1.$$

Interpret The tangent line to $f(x) = x^2$ at $x = 1$ has slope 2 and is parallel to the chord joining $A(0, 0)$ and $B(2, 4)$ (Figure 5.12). ***Now Try Exercise 1.***

EXAMPLE 2 Exploring the Mean Value Theorem

Explain why each of the following functions fails to satisfy the conditions of the Mean Value Theorem on the interval $[-1, 1]$.

(a) $f(x) = \sqrt{x^2} + 1$ **(b)** $f(x) = \begin{cases} x^3 + 3 & \text{for } x < 1 \\ x^2 + 1 & \text{for } x \ge 1 \end{cases}$

SOLUTION

(a) Note that $\sqrt{x^2} + 1 = |x| + 1$, so this is just a vertical shift of the absolute value function, which has a nondifferentiable "corner" at $x = 0$. (See Section 3.2.) The function f is not differentiable on $(-1, 1)$.

(b) Since $\lim_{x\to1^-} f(x) = \lim_{x\to1^-} x^3 + 3 = 4$ and $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} x^2 + 1 = 2$, the function has a discontinuity at $x = 1$. The function f is not continuous on $[-1, 1]$.

If the two functions given had satisfied the necessary conditions, the *conclusion* of the Mean Value Theorem would have guaranteed the existence of a number c in $(-1, 1)$ such that $f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)} = 0$. Such a number c does not exist for the function in part (a), but one happens to exist for the function in part (b) (Figure 5.13).

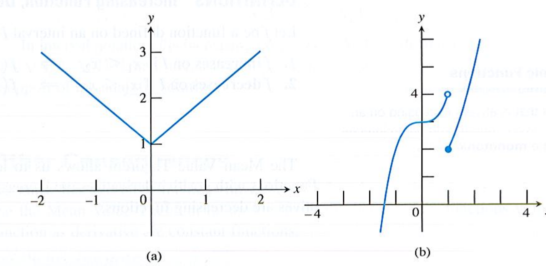


Figure 5.13 For both functions in Example 2, $\dfrac{f(1) - f(-1)}{1 - (-1)} = 0$ but neither function satisfies the conditions of the Mean Value Theorem on the interval $[-1, 1]$. For the function in Example 2(a), there is no number c such that $f'(c) = 0$. It happens that $f'(0) = 0$ in Example 2(b). ***Now Try Exercise 3.***

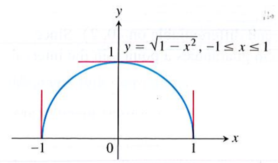


Figure 5.14 (Example 3)

EXAMPLE 3 Applying the Mean Value Theorem

Let $f(x) = \sqrt{1 - x^2}$, $A = (-1, f(-1))$, and $B = (1, f(1))$. Find a tangent to f in the interval $(-1, 1)$ that is parallel to the secant AB.

SOLUTION

The function f (Figure 5.14) is continuous on the interval $[-1, 1]$ and

$$f'(x) = \frac{-x}{\sqrt{1 - x^2}}$$

is defined on the interval $(-1, 1)$. The function is not differentiable at $x = -1$ and $x = 1$, but it does not need to be for the theorem to apply. Since $f(-1) = f(1) = 0$, the tangent we are looking for is horizontal. We find that $f' = 0$ at $x = 0$, where the graph has the horizontal tangent $y = 1$.

Now Try Exercise 9.

Physical Interpretation

If we think of the difference quotient $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that the instantaneous change at some interior point must equal the average change over the entire interval.

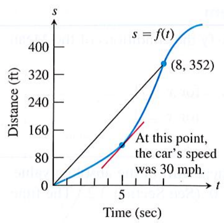


Figure 5.15 (Example 4)

EXAMPLE 4 Interpreting the Mean Value Theorem

If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ft/sec, or 30 mph. At some point during the acceleration, the theorem says, the speedometer must read exactly 30 mph (Figure 5.15).

Now Try Exercise 11.

Increasing and Decreasing Functions

Our first use of the Mean Value Theorem will be its application to increasing and decreasing functions.

DEFINITIONS Increasing Function, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. f **increases** on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
2. f **decreases** on I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Monotonic Functions

A function that is always increasing on an interval or always decreasing on an interval is said to be **monotonic** there.

The Mean Value Theorem allows us to identify exactly where graphs rise and fall. Functions with positive derivatives are increasing functions; functions with negative derivatives are decreasing functions.

COROLLARY 1 Increasing and Decreasing Functions

Let f be continuous on $[a, b]$ and differentiable on (a, b).

1. If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.
2. If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

Proof Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ gives

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2. The sign of the right-hand side of this equation is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore,

(a) $f(x_1) < f(x_2)$ if $f' > 0$ on (a, b) (f is increasing), or

(b) $f(x_1) > f(x_2)$ if $f' < 0$ on (a, b) (f is decreasing). ■

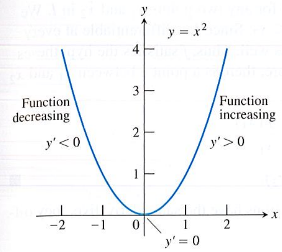


Figure 5.16 (Example 5)

EXAMPLE 5 Determining Where Graphs Rise or Fall

The function $y = x^2$ (Figure 5.16) is

(a) decreasing on $(-\infty, 0)$ because $y' = 2x < 0$ on $(-\infty, 0)$.

(b) increasing on $[0, \infty)$ because $y' = 2x > 0$ on $(0, \infty)$.

Now Try Exercise 15.

What's Happening at Zero?

Note that 0 appears in both intervals in Example 5, which is consistent both with the definition and with Corollary 1. Does this mean that the function $y = x^2$ is both increasing and decreasing at $x = 0$? No! This is because a function can only be described as increasing or decreasing on an *interval* with more than one point (see the definition). Saying that $y = x^2$ is "increasing at $x = 2$" is not really proper either, but you will often see that statement used as a short way of saying $y = x^2$ is "increasing on an interval containing 2."

EXAMPLE 6 Determining Where Graphs Rise or Fall

Where is the function $f(x) = x^3 - 4x$ increasing and where is it decreasing?

SOLUTION

Solve Analytically The function is increasing where $f'(x) > 0$.

$$3x^2 - 4 > 0$$

$$x^2 > \frac{4}{3}$$

$$x < -\sqrt{\frac{4}{3}} \quad \text{or} \quad x > \sqrt{\frac{4}{3}}$$

The function is decreasing where $f'(x) < 0$.

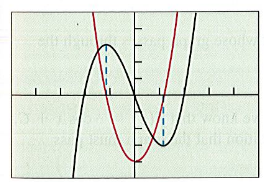
$$3x^2 - 4 < 0$$

$$x^2 < \frac{4}{3}$$

$$-\sqrt{\frac{4}{3}} < x < \sqrt{\frac{4}{3}}$$

In interval notation, f is increasing on $(-\infty, -\sqrt{4/3}]$, decreasing on $[-\sqrt{4/3}, \sqrt{4/3}]$, and increasing on $[\sqrt{4/3}, \infty)$. See Figure 5.17 for graphical support of the analytic solution.

Now Try Exercise 27.



[–5, 5] by [–5, 5]

Figure 5.17 By comparing the graphs of $f(x) = x^3 - 4x$ and $f'(x) = 3x^2 - 4$ we can relate the increasing and decreasing behavior of f to the sign of f'. (Example 6)

Other Consequences

We know that constant functions have the zero function as their derivative. We can now use the Mean Value Theorem to show conversely that the only functions with the zero function as derivative are constant functions.

COROLLARY 2 Functions with $f' = 0$ Are Constant

If $f'(x) = 0$ at each point of an interval I, then there is a constant C for which $f(x) = C$ for all x in I.

Proof Our plan is to show that $f(x_1) = f(x_2)$ for any two points x_1 and x_2 in I. We can assume the points are numbered so that $x_1 < x_2$. Since f is differentiable at every point of $[x_1, x_2]$ it is continuous at every point as well. Thus, f satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$. Therefore, there is a point c between x_1 and x_2 for which

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Because $f'(c) = 0$, it follows that $f(x_1) = f(x_2)$. ■

We can use Corollary 2 to show that if two functions have the same derivative, they differ by a constant.

COROLLARY 3 Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point of an interval I, then there is a constant C such that $f(x) = g(x) + C$ for all x in I.

Proof Let $h = f - g$. Then for each point x in I,

$$h'(x) = f'(x) - g'(x) = 0.$$

It follows from Corollary 2 that there is a constant C such that $h(x) = C$ for all x in I. Thus, $h(x) = f(x) - g(x) = C$, or $f(x) = g(x) + C$. ■

We know that the derivative of $f(x) = x^2$ is $2x$ on the interval $(-\infty, \infty)$. So, any other function $g(x)$ with derivative $2x$ on $(-\infty, \infty)$ must have the formula $g(x) = x^2 + C$ for some constant C.

EXAMPLE 7 Applying Corollary 3

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

SOLUTION

Since f has the same derivative as $g(x) = -\cos x$, we know that $f(x) = -\cos x + C$, for some constant C. To identify C, we use the condition that the graph must pass through $(0, 2)$. This is equivalent to saying that

$$\begin{aligned} f(0) &= 2 \\ -\cos(0) + C &= 2 \quad f(x) = -\cos x + C \\ -1 + C &= 2 \\ C &= 3. \end{aligned}$$

The formula for f is $f(x) = -\cos x + 3$. *Now Try Exercise 35.*

In Example 7 we were given a derivative and asked to find a function with that derivative. This type of function is so important that it has a name.

DEFINITION Antiderivative

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. The process of finding an antiderivative is **antidifferentiation.**

We know that if f has one antiderivative F then it has infinitely many antiderivatives, each differing from F by a constant. Corollary 3 says these are all there are. In Example 7, we found the particular antiderivative of $\sin x$ whose graph passed through the point $(0, 2)$.

EXAMPLE 8 Finding Velocity and Position

Find the velocity and position functions of a body falling freely from a height of 0 meters under each of the following sets of conditions:

(a) The acceleration is 9.8 m/sec² and the body falls from rest.

(b) The acceleration is 9.8 m/sec² and the body is propelled downward with an initial velocity of 1 m/sec.

SOLUTION

(a) ***Falling from rest.*** We measure distance fallen in meters and time in seconds, and assume that the body is released from rest at time $t = 0$.

Velocity: We know that the velocity $v(t)$ is an antiderivative of the constant function 9.8. We also know that $g(t) = 9.8t$ is an antiderivative of 9.8. By Corollary 3,

$$v(t) = 9.8t + C$$

for some constant C. Since the body falls from rest, $v(0) = 0$. Thus,

$$9.8(0) + C = 0 \quad \text{and} \quad C = 0.$$

The body's velocity function is $v(t) = 9.8t$

Position: We know that the position $s(t)$ is an antiderivative of $9.8t$. We also know that $h(t) = 4.9t^2$ is an antiderivative of $9.8t$. By Corollary 3,

$$s(t) = 4.9t^2 + C$$

for some constant C. Since $s(0) = 0$,

$$4.9(0)^2 + C = 0 \quad \text{and} \quad C = 0.$$

The body's position function is $s(t) = 4.9t^2$.

(b) ***Propelled downward.*** We measure distance fallen in meters and time in seconds, and assume that the body is propelled downward with velocity of 1 m/sec at time $t = 0$.

Velocity: The velocity function still has the form $9.8t + C$, but instead of being zero, the initial velocity (velocity at $t = 0$) is now 1 m/sec. Thus,

$$9.8(0) + C = 1 \quad \text{and} \quad C = 1.$$

The body's velocity function is $v(t) = 9.8t + 1$.

Position: We know that the position $s(t)$ is an antiderivative of $9.8t + 1$. We also know that $k(t) = 4.9t^2 + t$ is an antiderivative of $9.8t + 1$. By Corollary 3,

$$s(t) = 4.9t^2 + t + C$$

for some constant C. Since $s(0) = 0$,

$$4.9(0)^2 + 0 + C = 0 \quad \text{and} \quad C = 0.$$

The body's position function is $s(t) = 4.9t^2 + t$. *Now Try Exercise 43.*

Quick Review 5.2 *(For help, go to Sections 1.2, 2.3, and 3.2.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1 and 2, find exact solutions to the inequality.

1. $2x^2 - 6 < 0$ **2.** $3x^2 - 6 > 0$

In Exercises 3–5, let $f(x) = \sqrt{8 - 2x^2}$.

3. Find the domain of f.

4. Where is f continuous?

5. Where is f differentiable?

In Exercises 6–8, let $f(x) = \dfrac{x}{x^2 - 1}$.

6. Find the domain of f.

7. Where is f continuous?

8. Where is f differentiable?

In Exercises 9 and 10, find C so that the graph of the function f passes through the specified point.

9. $f(x) = -2x + C, \quad (-2, 7)$

10. $g(x) = x^2 + 2x + C, \quad (1, -1)$

Section 5.2 Exercises

In Exercises 1–8, **(a)** state whether or not the function satisfies the hypotheses of the Mean Value Theorem on the given interval, and **(b)** if it does, find each value of c in the interval (a, b) that satisfies the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

1. $f(x) = x^2 + 2x - 1$ on $[0, 1]$

2. $f(x) = x^{2/3}$ on $[0, 1]$

3. $f(x) = x^{1/3}$ on $[-1, 1]$

4. $f(x) = |x - 1|$ on $[0, 4]$

5. $f(x) = \sin^{-1} x$ on $[-1, 1]$

6. $f(x) = \ln(x - 1)$ on $[2, 4]$

7. $f(x) = \begin{cases} \cos x, & 0 \le x < \pi/2 \\ \sin x, & \pi/2 \le x \le \pi \end{cases}$ on $[0, \pi]$

8. $f(x) = \begin{cases} \sin^{-1} x, & -1 \le x < 1 \\ x/2 + 1, & 1 \le x \le 3 \end{cases}$ on $[-1, 3]$

In Exercises 9 and 10, the interval $a \le x \le b$ is given. Let $A = (a, f(a))$ and $B = (b, f(b))$. Write an equation for

(a) the secant line AB.

(b) a tangent line to f in the interval (a, b) that is parallel to AB.

9. $f(x) = x + \dfrac{1}{x}, \quad 0.5 \le x \le 2$

10. $f(x) = \sqrt{x - 1}, \quad 1 \le x \le 3$

11. ***Speeding*** A trucker handed in a ticket at a toll booth showing that in 2 h she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

12. ***Temperature Change*** It took 20 sec for the temperature to rise from 0°F to 212°F when a thermometer was taken from a freezer and placed in boiling water. Explain why at some moment in that interval the mercury was rising at exactly 10.6°F/sec.

13. ***Triremes*** Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 h. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots (sea miles per hour).

14. ***Running a Marathon*** A marathoner ran the 26.2-mi New York City Marathon in 2.2 h. Show that at least twice, the marathoner was running at exactly 11 mph.

In Exercises 15–22, use analytic methods to find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

15. $f(x) = 5x - x^2$ **16.** $g(x) = x^2 - x - 12$

17. $h(x) = \dfrac{2}{x}$ **18.** $k(x) = \dfrac{1}{x^2}$

19. $f(x) = e^{2x}$ **20.** $f(x) = e^{-0.5x}$

21. $y = 4 - \sqrt{x + 2}$ **22.** $y = x^4 - 10x^2 + 9$

In Exercises 23–28, find **(a)** the local extrema, **(b)** the intervals on which the function is increasing, and **(c)** the intervals on which the function is decreasing.

23. $f(x) = x\sqrt{4 - x}$ **24.** $g(x) = x^{1/3}(x + 8)$

25. $h(x) = \dfrac{-x}{x^2 + 4}$ **26.** $k(x) = \dfrac{x}{x^2 - 4}$

27. $f(x) = x^3 - 2x - 2\cos x$ **28.** $g(x) = 2x + \cos x$

In Exercises 29–34, find all possible functions f with the given derivative.

29. $f'(x) = x$ **30.** $f'(x) = 2$

31. $f'(x) = 3x^2 - 2x + 1$ **32.** $f'(x) = \sin x$

33. $f'(x) = e^x$ **34.** $f'(x) = \dfrac{1}{x - 1}, \quad x > 1$

In Exercises 35–38, find the function with the given derivative whose graph passes through the point P.

35. $f'(x) = -\dfrac{1}{x^2}, \quad x > 0, \quad P(2, 1)$

36. $f'(x) = \dfrac{1}{4x^{3/4}}, \quad P(1, -2)$

37. $f'(x) = \dfrac{1}{x + 2}, \quad x > -2, \quad P(-1, 3)$

38. $f'(x) = 2x + 1 - \cos x, \quad P(0, 3),$

Group Activity In Exercises 39–42, sketch a graph of a differentiable function $y = f(x)$ that has the given properties.

39. **(a)** local minimum at (1,1), local maximum at (3, 3)

(b) local minima at (1,1) and (3, 3)

(c) local maxima at (1,1) and (3, 3)

40. $f(2) = 3, \quad f'(2) = 0,$ and

(a) $f'(x) > 0$ for $x < 2$, $f'(x) < 0$ for $x > 2$.

(b) $f'(x) < 0$ for $x < 2$, $f'(x) > 0$ for $x > 2$.

(c) $f'(x) < 0$ for $x \ne 2$.

(d) $f'(x) > 0$ for $x \ne 2$.

41. $f'(-1) = f'(1) = 0, f'(x) > 0$ on $(-1, 1)$, $f'(x) < 0$ for $x < -1$, $f'(x) > 0$ for $x > 1$.

42. A local minimum value that is greater than one of its local maximum values.

43. ***Free Fall*** On the moon, the acceleration due to gravity is 1.6 m/sec².

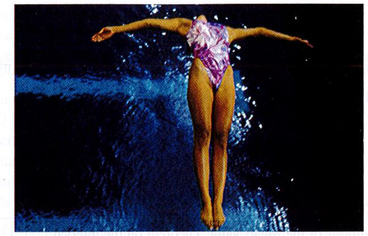
(a) If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

(b) How far below the point of release is the bottom of the crevasse?

(c) If instead of being released from rest, the rock is thrown into the crevasse from the same point with a downward velocity of 4 m/sec, when will it hit the bottom and how fast will it be going when it does?

44. ***Diving*** **(a)** With what velocity will you hit the water if you step off from a 10-m diving platform?

(b) With what velocity will you hit the water if you dive off the platform with an upward velocity of 2 m/sec?



45. **Writing to Learn** The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and at $x = 1$. Its derivative is equal to 1 at every point between 0 and 1, so f' is never zero between 0 and 1, and the graph of f has no tangent parallel to the chord from $(0, 0)$ to $(1, 0)$. Explain why this does not contradict the Mean Value Theorem.

46. **Writing to Learn** Explain why there is a zero of $y = \cos x$ between every two zeros of $y = \sin x$.

47. ***Unique Solution*** Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Also assume that $f(a)$ and $f(b)$ have opposite signs and $f' \ne 0$ between a and b. Show that $f(x) = 0$ exactly once between a and b.

In Exercises 48 and 49, show that the equation has exactly one solution in the interval. [*Hint:* See Exercise 47.]

48. $x^4 + 3x + 1 = 0, \quad -2 \le x \le -1$

49. $x + \ln(x + 1) = 0, \quad 0 \le x \le 3$

50. ***Parallel Tangents*** Assume that f and g are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between a and b where the tangents to the graphs of f and g are parallel or the same line. Illustrate with a sketch.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

51. **True or False** If f is differentiable and increasing on (a, b), then $f'(c) > 0$ for every c in (a, b). Justify your answer.

52. **True or False** If f is differentiable and $f'(c) > 0$ for every c in (a, b), then f is increasing on (a, b). Justify your answer.

53. Multiple Choice If $f(x) = \cos x$, then the Mean Value Theorem guarantees that somewhere between 0 and $\pi/3$, $f'(x) =$

(A) $-\dfrac{3}{2\pi}$ (B) $-\dfrac{\sqrt{3}}{2}$ (C) -1 (D) 0 (E) $\dfrac{1}{2}$

54. Multiple Choice On what interval is the function $g(x) = e^{x^3-6x^2+8}$ decreasing?

(A) $(-\infty, 2]$ (B) $[0, 4]$ (C) $[2, 4]$
(D) $(4, \infty)$ (E) no interval

55. Multiple Choice Which of the following functions is an antiderivative of $\dfrac{1}{\sqrt{x}}$?

(A) $-\dfrac{1}{\sqrt{2x^3}}$ (B) $-\dfrac{2}{\sqrt{x}}$ (C) $\dfrac{\sqrt{x}}{2}$
(D) $\sqrt{x} + 5$ (E) $2\sqrt{x} - 10$

56. Multiple Choice All of the following functions satisfy the conditions of the Mean Value Theorem on the interval $[-1, 1]$ *except*

(A) $\sin x$ (B) $\sin^{-1} x$ (C) $x^{5/3}$ (D) $x^{3/5}$ (E) $\dfrac{x}{x-2}$

Explorations

57. *Analyzing Derivative Data* Assume that f is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. The table gives some values of $f'(x)$.

x	$f'(x)$	x	$f'(x)$
−2	7	0.25	−4.81
−1.75	4.19	0.5	−4.25
−1.5	1.75	0.75	−3.31
−1.25	−0.31	1	−2
−1	−2	1.25	−0.31
−0.75	−3.31	1.5	1.75
−0.5	−4.25	1.75	4.19
−0.25	−4.81	2	7
0	−5		

(a) Estimate where f is increasing, decreasing, and has local extrema.

(b) Find a quadratic regression equation for the data in the table and superimpose its graph on a scatter plot of the data.

(c) Use the model in part (b) for f' and find a formula for f that satisfies $f(0) = 0$.

58. *Analyzing Motion Data* Priya's distance D in meters from a motion detector is given by the data in Table 5.1.

TABLE 5.1 Motion Detector Data

t (sec)	D (m)	t (sec)	D (m)
0.0	3.36	4.5	3.59
0.5	2.61	5.0	4.15
1.0	1.86	5.5	3.99
1.5	1.27	6.0	3.37
2.0	0.91	6.5	2.58
2.5	1.14	7.0	1.93
3.0	1.69	7.5	1.25
3.5	2.37	8.0	0.67
4.0	3.01		

(a) Estimate when Priya is moving toward the motion detector; away from the motion detector.

(b) **Writing to Learn** Give an interpretation of any local extreme values in terms of this problem situation.

(c) Find a cubic regression equation $D = f(t)$ for the data in Table 5.1 and superimpose its graph on a scatter plot of the data.

(d) Use the model in (c) for f to find a formula for f'. Use this formula to estimate the answers to (a).

Extending the Ideas

59. *Geometric Mean* The **geometric mean** of two positive numbers a and b is $\sqrt{ab}$. Show that for $f(x) = 1/x$ on any interval $[a, b]$ of positive numbers, the value of c in the conclusion of the Mean Value Theorem is $c = \sqrt{ab}$.

60. *Arithmetic Mean* The **arithmetic mean** of two numbers a and b is $(a + b)/2$. Show that for $f(x) = x^2$ on any interval $[a, b]$, the value of c in the conclusion of the Mean Value Theorem is $c = (a + b)/2$.

61. *Upper Bounds* Show that for any numbers a and b $|\sin b - \sin a| \le |b - a|$.

62. *Sign of f′* Assume that f is differentiable on $a \le x \le b$ and that $f(b) < f(a)$. Show that f' is negative at some point between a and b.

63. *Monotonic Functions* Show that monotonic increasing and decreasing functions are one-to-one.

5.3 Connecting f' and f'' with the Graph of f

What you will learn about . . .

- First Derivative Test for Local Extrema
- Concavity
- Points of Inflection
- Second Derivative Test for Local Extrema
- Learning About Functions from Derivatives

and why . . .

Differential calculus is a powerful problem-solving tool precisely because of its usefulness for analyzing functions.

First Derivative Test for Local Extrema

As we see once again in Figure 5.18, a function f may have local extrema at some critical points while failing to have local extrema at others. The key is the sign of f' in a critical point's immediate vicinity. As x moves from left to right, the values of f increase where $f' > 0$ and decrease where $f' < 0$.

At the points where f has a minimum value, we see that $f' < 0$ on the interval immediately to the left and $f' > 0$ on the interval immediately to the right. (If the point is an endpoint, there is only the interval on the appropriate side to consider.) This means that the curve is falling (values decreasing) on the left of the minimum value and rising (values increasing) on its right. Similarly, at the points where f has a maximum value, $f' > 0$ on the interval immediately to the left and $f' < 0$ on the interval immediately to the right. This means that the curve is rising (values increasing) on the left of the maximum value and falling (values decreasing) on its right.

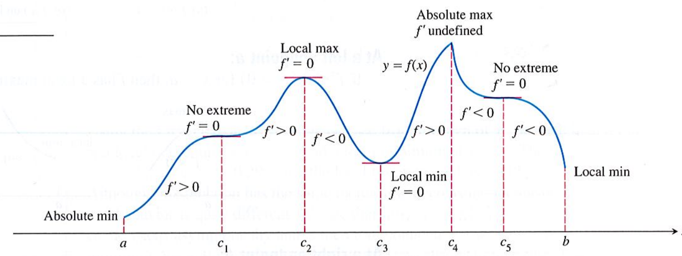


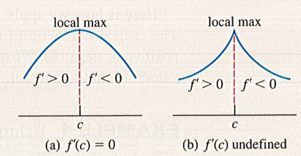
Figure 5.18 A function's first derivative tells how the graph rises and falls.

THEOREM 4 First Derivative Test for Local Extrema

The following test applies to a continuous function $f(x)$.

At a critical point c:

1. If f' changes sign from positive to negative at c ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then f has a local maximum value at c.



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